UDC 624.04:534.11

LINEAR MECHANICS OF SPIRAL PIPELINES

PMM Vol.43, No.3, 1979, pp. 569-574
B. A. GORDIENKO
(Kuibyshev)
(Received November 18, 1977)

Equations of motion are obtained for spatially-curvilinear elastic pipelines containing nonstationary flow of a viscous incompressible fluid. The influence of such factors as the rotational inertia and transverse shear strain of the pipe, the friction of the fluid on the internal pipeline surface, the pressure of the flow, is taken into account (the flow is characterized by parameters averaged over the cross-section). The problem is solved in a linear formulation under the assumption of nondeformability of the pipeline cross section

Feodos'ev [1] apparently first obtained the equation correctly describing the beam vibrations of elastic pipelines with an ideal fluid flow in a parabolic approximation. An analysis of this equation resulted in a conclusion about the existence of a critical flow velocity (V_*), above which the pipe loses the stability of the rectilinear equilibrium mode. An expression is found for V_*. Further investigations on the dynamics of straight and plane-curvilinear pipelines are contained in [2—10] (in the latter case, pipes bent into the arc of a circle were considered, as a rule, see [5, 6], etc. say). The behavior of systems under given laws of fluid motion is studied in [1—7], while the change in flow parameters for a given pipe motion (axial vibrations) is studied in [8]. The interrelated hydroelasticity problem, the axisymmetric vibrations of a "cylindrical shell-viscous fluid flow" system, is considered in [9, 10].

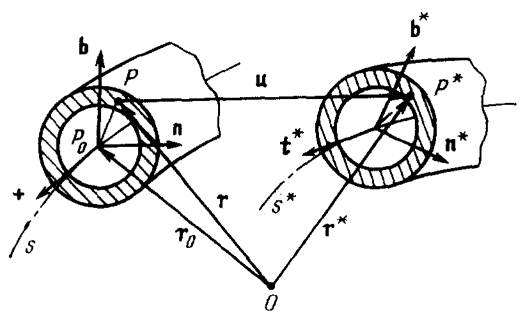

Fig. 1

The majority of the investigations mentioned was performed in a linear formulation. The exception is [2] in which the parametric tube vibrations were studied taking the geometric nonlinearity into account. The results of appropriate experiments are contained in [11, 12].

1. Mechanics of a pipeline. Let us note three points on a pipeline, P, P_0 and P^*, where P_0 is the projection of P on the axial line, and P^* is the position of P in the deformed state of the pipe (Fig. 1). Let $\mathbf{r}$, $\mathbf{r}_0$, $\mathbf{r}^*$ be the radius-vectors of these points from a common origin. If $\mathbf{u}$ is the displacement vector of the point P, then the following relationships are evident:

$$\mathbf{r}^* = \mathbf{r} + \mathbf{u}, \quad \mathbf{r} = \mathbf{r}_0 + \eta\mathbf{n} + \zeta\mathbf{b}, \quad \mathbf{u} = u\mathbf{t} + v\mathbf{n} + w\mathbf{b} \tag{1.1}$$

Here $\mathbf{t}, \mathbf{n}, \mathbf{b}$ are the directions of a natural trihedral referred to the undeformed state of the axial line of the pipe ($\mathbf{t} = d\mathbf{r}_0 / ds$); s, η, ζ are the appropriate coordinates of the point P.

Differentiating (1.1) we find

$$
\begin{aligned}
& d\mathbf{r}^* = (\partial \mathbf{r} / \partial \xi_i + e_{ij}\mathbf{e}_j)\, d\xi_i = \mathbf{e}_i^* (\eta, \zeta)\, d\xi_i \quad (d\mathbf{u} = e_{ij}\mathbf{e}_j d\xi_i) \\
& \mathbf{e}_1^* (\eta, \zeta) = (1 + e_{11} - k\eta)\, \mathbf{t} + (e_{12} - \varkappa\zeta)\, \mathbf{n} + (e_{13} + \varkappa\eta)\, \mathbf{b} \\
& \mathbf{e}_i^* (\eta, \zeta) = \mathbf{e}_i^* \; (i = 2, 3), \quad \mathbf{e}_i^* = (\delta_i^{\,j} + e_{ij})\, \mathbf{e}_j \quad (i = 1, 2, 3) \\
& (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \equiv \mathbf{t}, \mathbf{n}, \mathbf{b};\ \xi_1, \xi_2, \xi_3 \equiv s, \eta, \zeta) \\
& d\mathbf{r}^{*2} = g_{ij}^* d\xi_i d\xi_j, \quad d\mathbf{r}^2 = g_{ij} d\xi_i d\xi_j, \quad g_{ij}^* = g_{ij} + 2\varepsilon_{ij} \\
& 2\varepsilon_{ij} = e_{ij} + e_{ji} + e_{is}e_{js} - (1 + \delta_i^{\,j})\, a_{ij} = 2\varepsilon_{ji}, \quad a_{ij} = 0 \quad (i, j \neq 1) \\
& a_{1j} = (ke_{j1} - \varkappa e_{j3})\, \eta + \varkappa e_{j2}\zeta = a_{j1} \quad (j = 1, 2, 3) \\
& e_{11} = \partial u - kv, \quad e_{12} = \partial v + ku - \varkappa w, \quad e_{13} = \partial w + \varkappa v \\
& e_{ij} = \partial u_j / \partial \xi_i \; (i = 2, 3;\ j = 1, 2, 3;\ u_1, u_2, u_3 \equiv u, v, w;\ \partial \equiv \partial / \partial s)
\end{aligned}
$$

Here g_{ij}, g_{ij}^* are metric tensor components, $\delta_i^{\,j}$ is the Kronecker delta, $k, \varkappa$ are the curvature and torsion of the axial line, ε_{ij} are strain tensor components; unless especially stipulated, summation from one to three is understood by the repeated subscripts. The quadratic components $e_{is}e_{js}$ are not taken into account below.

The representations

$$
u = u_0 - \alpha\eta + \beta\zeta, \quad v = v_0 - \varphi\zeta, \; w = w_0 + \varphi\eta \tag{1.2}
$$

$$
\begin{aligned}
& e_{ij} = e_{ij,0} + e_{ij,1}\eta + e_{ij,2}\zeta, \quad \varepsilon_{ij} = \varepsilon_{ij,0} + \varepsilon_{ij,1}\eta + \varepsilon_{ij,2}\zeta \\
& e_{11,0} = u' - kv, \quad e_{12,0} = v' + ku - \varkappa w, \quad e_{13,0} = w' + \varkappa v \\
& e_{11,1} = -\alpha', \; e_{12,1} = -(k\alpha + \varkappa\varphi), \quad e_{13,1} = \varphi' \\
& e_{11,2} = \beta' + k\varphi, \quad e_{12,2} = -(\varphi' - k\beta), \quad e_{3,2} = -\varkappa\varphi \\
& e_{21,0} = -\alpha, \quad e_{22,0} = 0, \quad e_{23,0} = \varphi \\
& e_{31,0} = \beta, \quad e_{32,0} = -\varphi, \quad e_{33,0} = 0 \\
& e_{ij,k} = 0 \; (i = 2, 3;\ j = 1, 2, 3;\ k = 1, 2) \\
& \varepsilon_{11,0} = e_{11,0} = \varepsilon_{11}, \quad 2\varepsilon_{12,0} = e_{12,0} + e_{21,0} = 2\varepsilon_{12}, \\
& 2\varepsilon_{13,0} = e_{13,0} + e_{31,0} = 2\varepsilon_{13} \\
& \varepsilon_{11,1} = e_{11,1} - ke_{11,0} + \varkappa e_{13,0} = -\chi_{12}, \quad \varepsilon_{11,2} = e_{11,2} - \varkappa e_{12,0} = \chi_{13} \\
& 2\varepsilon_{12,1} = e_{12,1} - ke_{21,0} + \varkappa e_{23,0} = 0, \quad 2\varepsilon_{12,2} = e_{12,2} = -2\chi \\
& 2\varepsilon_{13,1} = e_{13,1} - ke_{31,0} = 2\chi, \quad 2\varepsilon_{13,2} = e_{13,2} - \varkappa e_{32,0} = 0 \\
& e_{22,k} = 0, \quad e_{33,k} = 0, \quad e_{23,k} = e_{32,k} = 0 \quad (k = 0, 1, 2)
\end{aligned} \tag{1.3}
$$

are used for the components $u_j, e_{ij}, \varepsilon_{ij}$. The prime here denotes differentiation with respect to the length $d\xi_1 \equiv ds$ of arc of the axial line, and the subscript zero in the components u_0, v_0, w_0 is omitted in (1.3).

It is seen from the relationships in the last line of (1.3) that a theory based on the expansions (1.2) will result in the consideration of a pipeline with an invariable outline which is in the uniaxial strain state. Hooke's law for this case can be written in the following dimensionless form

$$
\begin{aligned}
& \varepsilon_{11}^\circ = \varepsilon_{11}, \quad \varepsilon_{22}^\circ = \varepsilon_{33}^\circ = \mu\varepsilon_{11} / (1 - \mu), \quad \varepsilon_{ij}^\circ = 2k_0\varepsilon_{ij} \quad (i \neq j) \\
& \chi_{1j}^\circ = \chi_{1j} = m_{1j} / [\rho c^2 I_{j-1}] \; (j = 2, 3), \quad \chi^\circ = 2k_0\chi = m / [\rho c^2 I] \\
& \varepsilon_{ij}^\circ = \sigma_{ij} / [\rho c^2], \quad \rho c^2 = (1 - \mu)\, E / [(1 + \mu)(1 - 2\mu)]
\end{aligned}
$$

$$k_0 = (1 - 2\mu)\, k^\circ / [2\,(1 + \mu)]$$
$$I_{j-1} = \iint_F \xi_j^2\, dF \quad (j = 2, 3), \quad I = I_1 + I_2$$

Here σ_{ij}, ε_{ij}° are the physical stresses and their dimensionless analogs, m_{1j}, m are the bending moment and torque, I_k, I are the section moments of inertia, axial and under torsion, E, μ, ρ are the Young's modulus, Poisson's ration, and density of the material, c is the speed of sound, k° is the coefficient of the tangential stress distribution over the pipe section.

Expressions for the kinetic (T) and potential (U) energies of pipelines are written as follows in dimensionless form:

$$T = \frac{1}{2}\int_0^1 \left(\frac{1}{F}\iint_F \mathbf{u}^{\cdot 2} dF\right) ds, \quad U = \frac{1}{2}\int_0^1 \left(\frac{1}{F}\iint_F \varepsilon_{ij}\varepsilon_{ij}^\circ\, dF\right) ds \tag{1.4}$$

Here F is the cross-sectional area of the pipe; the dots denote differentiation with respect to the dimensionless time $t = cT_0 / L$ (T_0 is the physical time). The linear quantities are referred to the length of the axial line L and the velocity to the speed of sound c.

2. Fluid mechanics. In this paper the fluid is considered viscous and incompressible. The flow is characterized by parameters averaged over the section. If $\mathbf{u}_0$ is the displacement vector of axial points of the pipeline, then

$$\mathbf{V} = \mathbf{u}_0^{\cdot} + V\mathbf{t}^*, \quad \mathbf{u}_0^{\cdot} = u_j^{\cdot}\mathbf{e}_j\, (u_j^{\cdot} \equiv u_{j,0}^{\cdot}), \quad \mathbf{t}^* = (\delta_1^{\,j} + e_{1j})\,\mathbf{e}_j \tag{2.1}$$
$$T_1 = \frac{1}{2}\rho_1 F_1 \int_0^1 (\mathbf{u}^{\cdot} + V\mathbf{t}^*)^2\, ds$$

Here $\mathbf{u}_0^{\cdot}$ is the transfer velocity vector, V, $\mathbf{V}$ are the fluid velocity relative to the pipe and the absolute velocity vector (V, $\mathbf{V}$, $\mathbf{u}_0^{\cdot}$ are in units of c), ρ_1 is the fluid density in units of ρ, and F_1 is the area of an inner pipe section in units of F.

The work of the friction and pressure forces on the virtual displacement $\delta\mathbf{u}_0$ can be taken into account by the expression

$$\delta A = \int_0^1 (\mathbf{P} + \mathbf{Q})\, \delta\mathbf{u}_0\, ds \tag{2.2}$$

$$\mathbf{P} = p \oint_C \frac{d\mathbf{S}^*}{ds}, \quad \mathbf{Q} = \tau\rho_1 F_1 V^2 \mathbf{t}^* \quad \left(\tau = \frac{\nu}{8r}\right) \tag{2.3}$$
$$d\mathbf{S}^* / ds = d\mathbf{R}^* \times \mathbf{t}^*(\eta, \zeta) = (X_j d\eta + Y_j d\zeta)\,\mathbf{e}_j$$
$$d\mathbf{R}^* = \mathbf{n}^* d\eta + \mathbf{b}^* d\zeta = [(\delta_2^{\,j} + e_{2j})\, d\eta + (\delta_3^{\,j} + e_{3j})\, d\zeta]\,\mathbf{e}_j$$
$$\mathbf{t}^*(\eta, \zeta) = (1 + e_{11} - k\eta)\,\mathbf{t} + (e_{12} - \varkappa\zeta)\,\mathbf{n} + (e_{13} + \varkappa\eta)\,\mathbf{b}$$
$$X_1 = e_{13} + \varkappa\eta + \varkappa e_{23}\zeta, \quad Y_1 = -e_{12} + \varkappa e_{32}\eta + \varkappa\zeta$$
$$X_2 = e_{23} - (ke_{23} + \varkappa e_{21})\,\eta, \quad Y_2 = (1 + e_{11}) - (k + \varkappa e_{31})\,\eta$$
$$X_3 = -(1 + e_{11}) + k\eta - \varkappa e_{21}\zeta, \quad Y_3 = -e_{32} + ke_{32}\eta - \varkappa e_{31}\zeta$$

Here $\mathbf{Q}$, $\mathbf{P}$ are linear friction and pressure forces, ν, r are the drag coefficients in the Darcy—Weissbach formula and the hydraulic radius (see [13], for instance), p is the pressure in the flow (in units of ρc^2), dS^* is the area of a pipeline area element oriented in the direction of the external normal, $\mathbf{R}^*$ is the radius-vector of the point P^* in the system $\mathbf{e}_i^*$, C is the outline of a section drawn through P^*, and $d\mathbf{R}^*$ is a vector in a direction tangent to the contour C at the point P^*. All quantities are referred to the inner surface of the deformed pipe.

Using the Green's formula and taking into account that in a linear approximation X_j and Y_j depend linearly on η, ζ, we obtain

$$\mathbf{P} = p \oint_C (X_j d\eta + Y_j d\zeta)\, \mathbf{e}_j = p \iint_{F_1} \left(\frac{\partial Y_j}{\partial \eta} - \frac{\partial X_j}{\partial \zeta}\right) \mathbf{e}_j\, d\eta\, d\zeta = pF_1 P_j \mathbf{e}_j \tag{2.4}$$

$$P_j = (-1)^{j-1}(k\delta_2{}^j + \varkappa_j), \quad \varkappa_1 = k\alpha, \quad \varkappa_2 = \alpha' + \varkappa\beta, \quad \varkappa_3 = \beta' - \varkappa\alpha$$

Taking account of (2.3) and (2.4), the expression for the virtual work (2.2) becomes

$$\delta A = \rho_1 F_1 \int_0^1 \left[(\delta_1{}^j + e_{1j})\, \tau V^2 + (-1)^{j-1}(k\delta_2{}^j + \varkappa_j)\, \frac{p}{\rho_1}\right] \delta u_j\, ds \tag{2.5}$$

3. Equations of motion. Let us use the Hamilton—Ostrogradskii principle

$$\delta \int_{t_0}^{t} (T + T_1 - U)\, dt + \int_{t_0}^{t} \delta A\, dt = 0 \tag{3.1}$$

Substituting (1.4), (2.1), (2.5) into (3.1) and integrating by parts, we obtain

$$\frac{1}{2} \int_{t_0}^{t} \int_0^1 \left[\frac{\partial H}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial H}{\partial u^{\cdot}}\right) - \frac{\partial}{\partial s}\left(\frac{\partial H}{\partial u'}\right) + 2A\right] \delta u\, ds\, dt +$$
$$\frac{1}{2} \int_0^1 \left(\frac{\partial H}{\partial u^{\cdot}}\, \delta u\right)\Big|_{t_0}^{t} ds + \frac{1}{2} \int_{t_0}^{t} \left(\frac{\partial H}{\partial u'}\, \delta u\right)\Big|_0^1 dt = 0 \tag{3.2}$$

$$H = j_k{}^2 (u_{j,k}^{\cdot 2} - \varepsilon_{ij,k} \overset{\circ}{\varepsilon}_{ij,k}) + \rho_1 F_1 [u_{j,0}^{\cdot} + (\delta_1{}^j + e_{1j})\, V]^2, \quad u \equiv u_{j,k}$$

$$A \equiv A_{j,k} = \delta_0{}^k \rho_1 F_1 [(\delta_1{}^j + e_{1j})\, \tau V^2 + (-1)^{j-1}(k\delta_2{}^j + \varkappa_j)\, p / \rho_1]$$

$$j_k{}^2 = I_k / (FL^2) = 1 / \lambda_k{}^2, \quad L \equiv 1 \ (i, j = 1, 2, 3;\ k = 0, 1, 2)$$

Here λ_k, j_k $(k = 1, 2)$ are the pipe flexibility and stability parameters relative to the axes $\mathbf{b}$ and $\mathbf{n}$, respectively. Because of the arbitrariness of the variation $\delta u_{j,k}$ it follows from (3.2) that

$$\partial (\partial H / \partial u^{\cdot}) / \partial t + \partial (\partial H / \partial u') / \partial s = \partial H / \partial u + 2A \tag{3.3}$$

$$[(\partial H / \partial u^{\cdot})\, \delta u]_{t_0}{}^t = 0, \quad [(\partial H / \partial u')\, \delta u]_0{}^1 = 0 \quad (u = u_{j,k}) \tag{3.4}$$

In the case under consideration, (3.3) (the Euler—Ostrogradskii equations) are the desired equations of motion for a pipeline in a fluid flow. The expressions (3.4) determine the initial and boundary conditions of the problem. Using the relationships obtained, we represent the equations of motion in the following form (it is assumed

$j_1 = j_2 = j$) below):

$$
\begin{aligned}
&u^{\cdot\cdot} = \sigma(\partial\varepsilon_{11}{}^\circ - k\varepsilon_{12}{}^\circ + j^2 b_1) - (1-\sigma) q_1 \\
&v^{\cdot\cdot} = \sigma(\partial\varepsilon_{12}{}^\circ + k\varepsilon_{11}{}^\circ - \varkappa\varepsilon_{13}{}^\circ + j^2 b_2) - (1-\sigma) q_2 \\
&w^{\cdot\cdot} = \sigma(\partial\varepsilon_{13}{}^\circ + \varkappa\varepsilon_{12}{}^\circ + j^2 b_3) - (1-\sigma) q_3 \\
&\alpha^{\cdot\cdot} = \partial\chi_{12}{}^\circ + \lambda^2\varepsilon_{12}{}^\circ, \quad \beta^{\cdot\cdot} = \partial\chi_{13}{}^\circ - \lambda^2\varepsilon_{13}{}^\circ \\
&\varphi^{\cdot\cdot} = \partial\chi^\circ - k\chi_{13}{}^\circ / 2 \quad (\sigma = 1/(1+\rho_1 F_1)) \\
&b_1 = \partial(k\chi_{12}{}^\circ) + k\varkappa\chi_{13}{}^\circ, \quad b_3 = -\partial(\varkappa\chi_{12}{}^\circ) - \varkappa^2\chi_{13}{}^\circ \\
&b_2 = -\partial(\varkappa\chi_{13}{}^\circ) + (k^2+\varkappa^2)\chi_{12}{}^\circ \\
&q_1 = V_1{}^{\cdot} - u^{\cdot\cdot} + (VV_1)' - kVV_2 + (1+e_{11})\tau V^2 + \varkappa_1 p/\rho_1 \\
&q_2 = V_2{}^{\cdot} - v^{\cdot\cdot} + (VV_2)' + kVV_1 - \varkappa VV_3 + e_{12}\tau V^2 - (k+\varkappa_2) p/\rho_1 \\
&q_3 = V_3{}^{\cdot} - w^{\cdot\cdot} + (VV_3)' + \varkappa VV_2 + e_{13}\tau V^2 + \varkappa_3 p/\rho_1 \\
&V_j = u_j{}^{\cdot} + (\delta_1{}^j + e_{1j}) V, \quad u_j \equiv u_{j,0}, \quad e_{ij} \equiv \varepsilon_{ij,0}
\end{aligned}
\tag{3.5}
$$

Relationships governing the fluid motion in the pipe must be appended to the pipeline equations of motion to close the system (3.5). In the approximation under consideration, the following linearized equations

$$
-\frac{\partial p}{\partial t} = \rho_1 c_1^2 \frac{\partial V}{\partial s}, \quad -\frac{\partial p}{\partial s} = \rho_1\left(\frac{\partial V}{\partial t} + 2aV\right) \tag{3.6}
$$

for instance, can be used for this. Here c_1 is the speed of sound in the fluid taking elasticity of the pipeline into account (in units of c), and a is a hydraulic drag parameter. For a laminar flow $2a \approx \tau V \approx \mathrm{const}$.

When solving specific problems, (3.5) and (3.6) should be supplemented by the necessary number of initial and boundary conditions. Taking into account that the fluid and pipeline in the model constructed above are considered as spatially one-dimensional objects, these conditions can be written as follows in the general case:

$$
\begin{aligned}
&\psi(s, 0) = \psi_0(s), \quad \psi^{\cdot}(s, 0) = \psi_0{}^{\cdot}(s) \\
&\psi(0, t) = \psi_1(t), \quad \psi(1, t) = \psi_2(t) \\
&(\psi \equiv u, v, w, \alpha, \beta, \varphi, p, V)
\end{aligned}
$$

Here ψ_j ($j = 0, 1, 2$) are given functions of the arguments mentioned.

Therefore, linear equations of motion have been obtained for spatially curvilinear elastic pipelines of constant cross-section and arbitrary geometry, which contain a viscous incompressible fluid flow. These equations belong to the hyperbolic type and may consequently be used not only to study ordinary classical vibrations, but also to analyze wave processes associated with the nonstationary deformation of pipelines. The approach followed above evidently allows construction of more refined theories.

REFERENCES

1. Feodos'ev, V. I., On pipe vibrations and stability during fluid flow through it, Inzhen. Sb., Vo. 10, 1951.

2. Natanzon, M. S., Parametric vibrations of a pipeline excited by a pulsating fluid discharge, Izv. Akad. Nauk SSSR, OTN, Mekhan. i Mashinostr., No. 4, 1962.

3. R o t h, V. W., Instabilität durchströmter Rohre, Ingr.-Arch. Vo. 33, No. 4, 1964. pp. 236-263.

4. M o v c h a n, A. A., On a problem of stability of a pipe with a fluid, flowing through it. PMM, Vol. 29, No. 4, 1965.

5. K o v r e v s k i i, A. P. Dynamics of pipelines containing an unsteady fluid flow Priklad. Mekhan., Vol. 6, No. 8, 1970.

6. U n n y, T. E., M a r t i n, E. L., and D u b e y, R. N., Hydroelastic instability of uniformly curved pipe-fluid systems. Trans. ASME, Ser. E. J. Appl. Mech. Vol. 37, No. 3, 1970.

7. S t e i n, R. A., and T o b r i n e r, M. W., Vibration of pipes containing flowing fluids. Trans. ASME, Ser. E. J. Appl. Mech., Vol. 37, No. 4, 1970.

8. A r o n o v i c h, G. V., Motion of a viscous fluid in a longitudinally vibrating pipe, PMM, Vol. 8, No. 1, 1944,

9. G u b e n k o, V. S., D e r k a c h, P. Kh., and K u z n e t s o v, V. N., Nonstationary one-dimensional viscous fluid flow in a deformable pipe, Prikl. Mekhan., Vol. 9, No. 4, 1973.

10. G u b e n k o, V. S. and K u z n e t s o v, V. N., Axisymmetric viscous fluid flow within an elastic cylindrical pipe. Prikl. Mekhan., Vol. 12, No. 8, 1976.

11. B e n j a m i n, T. B., Dynamics of a system of articulated pipes conveying fluid. II. Experimental, Proc. Roy. Soc. A, Vol. 261, No. 1307, p. 487-499, 1961.

12. G r e g o r y, R. W. and P a i d o u s s i s, M. P., Unstable oscillation of tubular cantilevers conveying fluid. II. Exptl. Proc. Roy. Soc. A, Vol. 293, No. 1435, p. 528-542, 1966.

13. C h a r n y i, I. A., Unsteady Motion of a Real Fluid in Pipes. "Nedra", Moscow, 1975.

Translated by M. D. F.